

***FOUNDATIONS OF GEOMETRY IN
ITALY BEFORE HILBERT***

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If things had gone according to plan, it should have been an Italian to voice the new ideas since no one had come closer to those ideas than the [Italian] school.

... In fact it was also an Italian. I realised that only after having completed this essay. Fano had already got there in 1892. He introduces his axiomatic system with words that resonate with Hilbert's own words that we have quoted above. (Freudenthal 1957)

This quotation shows that Italian school had elaborated a point of view on foundations of geometry resonant with Hilbert's about ten years before the publication of Hilbert's work.

In this talk I will try to reconstruct the link between the Italian geometric tradition on the foundations of geometry, from the work of De Paolis (1881) to the first appearance of Hilbert's masterpiece (1899).

Before beginning, I give a short list of the main features of Italian school:

Riccardo De Paolis (1854 – 1892) student of Luigi Cremona (1830 – 1903).

Corrado Segre (1860 – 1924)

Giuseppe Veronese (1854 – 1920)

Gino Fano (1870 – 1940)

Federigo Enriques (1870 – 1946)

Giuseppe Vailati (1863 – 1909)

Mario Pieri (1860 – 1913)

Giuseppe Peano (1858 – 1932)

1. Riccardo De Paolis

His only paper on foundations of geometry is: **Sui fondamenti della geometria proiettiva**, *Atti della R. Acc. dei Lincei*, 1882

De Paolis' works stems directly from von Staudt's program shown in his *Geometrie der Lage*, namely: to make projective geometry free from metrical properties.

In this context the influence of Felix Klein's *Über die sogenannte Nicht-Euklidische Geometrie*, *Math. Ann*, 1870 – 72, is very strong:

For De Paolis, in the same way as for Klein *it is vital to note that projective geometry can be developed before solving the problem of metric determination*, so his ultimate aim is to

obtain the coordinatization of the projective space on the basis of purely geometric considerations.

De Paolis follows mainly Möbius' idea of *rationality net*:

De Paolis, as von Staudt, starts from the fourth harmonic of three points A, B, C in a line; he calls harmonic system the minimal set of points in the line closed with respect to this ternary operation. It is easy to prove that an harmonic set is isomorphic with rational numbers. To extend this isomorphism to real numbers he needs some "continuity" principle to complete the harmonic system. Therefore he introduces a new postulate similar to Dedekind's.

The introduction of these elements is similar to the introduction of irrational numbers in arithmetic. The aim is to permit constructions and considerations which would otherwise be impossible.

The problem of giving a sound basis to projective geometry became more and more urgent for Italian geometers in the because of their growing recourse to hyperspatial methods.

In this context their approach was heavily influenced by the work of Plücker and Grassmann.

2. Plücker and Grassmann.

There is an intimate connection between the work of Grassmann and Plücker. The former developed a part of n -dimensional pure geometry, the latter showed how to regard so-called three-dimensional space, for example, as an n -dimensional manifold with respect to certain elements.

Thus, a space whose elements are Plücker's line-complexes turns out to have the same formal properties as Grassmann's five-dimensional spaces. And conversely, each n -dimensional space of Grassmann may be interpreted in terms of the usual three-dimensional space when certain appropriate geometrical configurations are taken as the elements. (Nagel)

As it is well known, Plücker, in his influential paper of 1868 had clearly explored the idea that a line in the geometry of ordinary three-space may be interpreted as a point in a four-dimensional space. Ordinary ruled space is therefore considered as a four dimensional space, whose elements are no longer points, but lines.

This fact has two important foundational implications: on the one hand dimension of space is not anymore absolute; on the other hand the generating element of space, the point, was almost definitively losing its usual meaning.

In Italy this point of view was deeply understood by a young mathematician in Turin, Corrado Segre.

3. Segre's thesis 1884

Segre was only twenty when he got his degree. He begins his thesis in a very bold way:

The geometry of n -dimensional spaces has now found its place amongst the branches of mathematics. And even if we consider it aside from the important applications to ordinary geometry, that is, even when the element or point of that space is not considered as a geometrical element (and not even as an analytical element made of values of n variable quantities), but as an element in itself, whose intimate nature is left undetermined, it is impossible not to acknowledge the fact that it is a science, in which all propositions are rigorous, since they are obtained with essentially mathematical reasoning. The lack of a representation for our senses does not matter greatly to a pure mathematician.

Born, as it were, out of Riemann's famous work of 1854 ... n -dimensional geometry develops along two separate lines: the first one deals with the curvature of spaces and is therefore connected with non-Euclidean geometry, the second one studies the projective geometry of linear spaces ... and in my work I am to focus on the latter. This path opens for keen mathematicians an unbound richness of extremely interesting research.

In this way Segre, since the very beginning, poses the point *as an element in itself, whose intimate nature is left undetermined*. We may here clearly understand the origins of Fano's point of view which had surprised Freudenthal. Any link with sensible reality was already cut in 1884: *The lack of a representation for our senses does not matter greatly to a pure mathematician*.

According to this conception the new object of projective geometry is no longer "real" space, but what we now call a vector space. Segre tries to give a definition of linear space:

Any continuous set of elements, whose number is m -fold infinite will form an m -dimensional space formed by such elements. ... any m -dimensional space is considered linear when to each of its elements it is possible to attribute the numeric values (real or imaginary ones) of m quantities, so that, with no exception, to each arbitrary group of values of such quantities corresponds only one element of that space, and vice versa.

These values are said coordinates of the element. If we represent them with the ratios of m other quantities to one $(m+1)$ -th, these will constitute the $m+1$ homogenous coordinates of the element of the space considered, so that each element of this space, without exception, will be identified by the mutual ratios of these homogeneous coordinates, and vice versa it will identify their own ratios.

And he continues giving a definition of isomorphism:

All linear spaces with the same number of dimensions can be considered identical between them, because in their study we are not considering the nature of those elements, but only the property of linearity and the number of dimensions of the space formed by those very elements. Therefore we will be able to apply, for example, the theory of projectivity, of harmonic groups, involution theory, etc, in the forms of first kind.

We will see that Segre's definition was harshly criticized by Pean, but, despite of the lack of rigour his point of view was very open to further developments. Indeed his approach led to numerous foundational developments, and in a few years he would submit these developments to his best students as research topics.

4. Segre's programme (1891)

As far as I know no one has yet identified and discussed a system of independent postulates that characterise the n -dimensional linear space, from which the representation of its points by coordinates can be derived. It would be very useful if some students wanted to make of this issue their research topic (which does not seem difficult).

These words are written in the notes of Segre's course in 1891, which dealt with the "Introduction to geometry of simply infinite algebraic entities" and show a research program which was took over by Fano who attended to this course.

Segre's programme indeed aimed at demonstrating the possibility of inferring coordinates from a set of independent postulates. His programme was therefore the natural consequence of Staudt's and De Paolis' approach and represents a main thread in the developing of Italian school.

5. Peano's point of view (1889)

In the meantime Peano, who criticized the many imperfections of Segre's definition of linear space (e. g. he had not excluded from the coordinates of projective hyperspaces the n -ple $(0,0,0, \dots, 0)$), elaborated, on the basis of Grassmann's *Ausdehnungslehre* his own definition of n -dimensional real linear space, which is almost identical with our definition.

We have therefore a category of entities called points. These entities are not defined. Furthermore, given three points, we consider a relation between them, which we indicate with the expression $c \in ab$. This relation is not defined either. The reader can understand by the sign l any category of elements, and by $c \in ab$ any relation between three elements of that category.

All the following definitions will always be true ... If a certain group of axioms is true, then all the deduced propositions will also be true.

I don't cite *in extenso* Peano's definition which is, as I already said, the same as our own.

It is important to compare the two definitions briefly, also in the light of the debate that shortly ensued. Both definitions were generated by a rethinking of Grassmann's ideas, and both aimed at establishing a solid foundation for the basic concepts of geometry. The difference in rigour between the two definitions is only too clear, and therefore we will not mention it any further. However, I may notice that in Peano's case the definition, placed as it is at the end of his work, is more of a synthesis of his preceding work than a basis on which to build a new geometric theory. Segre's position is quite different. He placed his definition at the very beginning of his dissertation, and in so doing he indicated it as the conceptual basis of the projective geometry of hyperspaces, which is in its turn the conceptual basis for the developments of algebraic geometry.

6. Van der Waerden commentary (1986)

Ce point de vue logique tout à fait abstrait est très remarquable. Dans les exposés des géomètres allemands il est dit souvent que ce point de vue est dû à Hilbert, mais c'est Peano qui a exposé, le premier, ce point de vue logique.

It is probably worth noticing that van der Waerden here remarks that Peano had reached this abstract point of view well before Hilbert, and at least four years before the date that Freudenthal ascribes to Fano. As a matter of fact, however, Segre had already five years earlier turned such point of view into an "ordinary working tool for Italian geometers".

7. Veronese 1884 and 1891

Giuseppe Veronese published his ideas about hyperspaces in 1882, in an article in the *Mathematische Annalen*, but he developed them in depth only in 1891 in a massive volume (*Foundations of Geometry*) I do not intend to go into too many details regarding Veronese's point of view, which, as it is well known, had also some influence on Hilbert (the *Foundations* were translated into German in 1894). Veronese was the first to introduce the concept of a non-Archimedean geometry – the prototype, after non-Euclidean geometry, of the long series of *non* geometries that characterised the foundations of geometry in the 20th century.

For Veronese too, the basic idea of geometry of hyperspaces is born out of the necessity of using a more powerful tool for the synthetic study of the geometrical objects of ordinary

space. Conics, for instance, are typically studied in a more efficient and easier way as projections of the circle (or, which is the same thing – as sections of a cone). Similarly, the study of complicated surfaces in ordinary space can be simplified by considering them as projections or sections of hypersurfaces. It is in this framework that in 1884 we see the appearance (thanks to Veronese and Segre independently) of what is probably the most famous hypersurface in the history of mathematics: the *varietà di Veronese*, a four-dimensional surface in a five-dimensional space, amongst whose sections and projections some of the most important surfaces of ordinary space are to be found (for example the so-called *Roman* surface studied by Steiner).

The method of projection and section introduced by Segre and Veronese obtained very important results, and it was helpful in clarifying and systematising a great number of results by providing new techniques. Veronese's work of 1891, with its exceptional timing with respect to Segre's research programme announced in the same year, expressed some teaching projects in a somewhat confused way, as well as trying to give conceptual, axiomatic and almost philosophical conceptual foundations to an *instrument* that had already proved to be highly effective for the study of definite problems.

The charm of Veronese's text is the combination of philosophical observations on the nature of geometry with practical and foundational problems of mathematics. However, this is also a limitation. On the one hand his approach is extremely advanced, but on the other his presentation is very old fashioned. It is almost impossible, for instance, to find a clearly defined pattern in the axioms he proposes. They are divided into the geometrical and the practical, and they alternate with philosophical observations that certainly do not help to clarify the presentation. However, despite the difficulty of the style, it is possible to identify some central points that justify the admiration shown to this work.

In any case Veronese does not think about points in hyperspace as numbers or as objects in the ordinary space (in Plücker's way):

Here the point is not defined as a system of numbers, nor as an object of whatever nature, but as the point exactly how we imagine it in ordinary space; and objects made of points are objects (figures) to which we continually apply both spatial intuition and abstraction, and therefore the synthetic method.

Veronese, who had been one of Klein's students, was still anchored to the empiricist conception of geometry that had been at the forefront of research in the two decades that preceded his work. Therefore, just like Pasch, Veronese attributed an empirical nature to geometrical axioms, without contradicting the abstract nature of the discipline:

Geometry is the most exact experimental science, because the objects outside thought, that we need for the formulation of axioms, are replaced in our mind by abstract forms, and therefore the truths of the objects can be demonstrated with the combination of the forms independently obtained from what happens outside

But such purely empiricist conception was definitely not enough for Veronese and it did not allow him to organise the rich material that he has himself produced in a conceptual way. Experience provides, as it were, only the nucleus of the system of axioms. In the case of arithmetic, the study of real numbers led “naturally”, so to speak, to the study of complex numbers, which are indispensable for the study of “real” problems independently of their existence - or their non-existence - “outside thought”. Similarly in geometry it is correct to apply “empirical” axioms to “imaginary” objects in order to create new and powerful tools for the study of concrete problems - just as the hyperspaces had turned out to be powerful tools for the study of ordinary space. Such expansion of the system of axioms is achieved, in perfect pre-Hilbert style, in a totally arbitrary way. The only principle taken into account is the principle of non-contradiction:

In the field of mathematics it is possible to have a well determined definition, a postulate, or a hypothesis, whose terms do not contradict each other and do not contradict the principles, the operations and the truths from which they are derived ... A hypothesis is mathematically false only when it establishes a property that is or that can be demonstrated to be in contradiction with the preceding truths, or with those that can be inferred from them Once the characteristics of the mathematical forms have been established, mathematical possibility is regulated by the principle of non-contradiction ... And a possibility becomes a mathematical reality, albeit an abstract one.

8. Peano's criticism 1891

The dispute between Segre and Veronese and Peano broke out shortly after and was extremely ferocious. The main point of contention was the question of *rigour*, but we must not be led to believe it was the only one. The reasons of the three opposing parties are easily understood. Every one was involved in important and far-reaching scientific studies, to which they were dedicating all their energies. Peano had achieved extraordinary results in his effort to give rigour to the fundamental body of mathematics of his time, and he was also receiving international recognition in this field. Any compromise on the issue of rigour could not but appear a compromise with the old and careless way of “doing mathematics”.

On the other hand Segre and Veronese envisioned the foundations of a building that was both new and fascinating, and in comparison an excessive request for rigour seemed like tying one's hands behind one's back and give up the challenge of climbing peaks. In this

perspective the two approaches mirrored necessities that were both real and deep, and that were destined to be constantly debated in years to come. But there are other aspects to consider. Peano was still very much linked to an empiricist conception of mathematics. Mathematics is “a perfected logic” and therefore:

each author is allowed to accept the experimental laws that more appeal to him, and he can also propose any hypothesis he likes ... [but] ... If an author starts from hypotheses that are contrary to experience, or from hypotheses that cannot be verified by experience, nor their consequences, he will be able, for sure, to infer some wonderful theory that will lead other to cry: what gain if the author had applied his reasoning to practical hypotheses!

On this basis Peano could not accept hyperspace geometry and non-Archimedean geometry, not only because he considered them less rigorous (if this was the only obstacle, he could have endeavoured to make them more rigorous), but because he thought them “useless”! Obviously, such an attitude would not contribute a great deal to the study of the foundations of geometry. Mario Pieri, to whom we will come back later in this article, succeeded in amalgamating Peano’s rigour with the intellectual audacity and the pre-formalism of both Veronese and Segre.

9 Fano's approach 1892

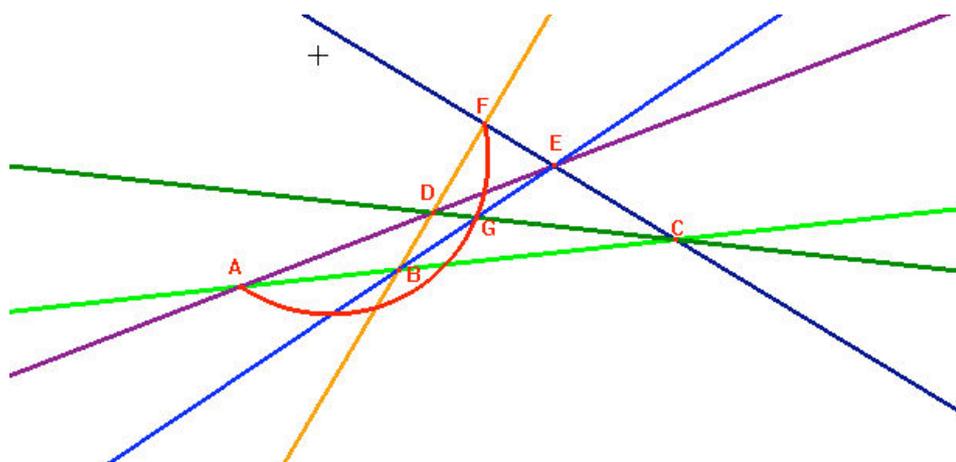
A manifold [set] of entities of whatever nature; entities that, for brevity's sake, we will call points, however we are obviously leaving out any consideration of their very nature. ... I even prefer to keep the definition of postulates [... for those properties that] will give us the prime properties of the entities or points of our manifold; those (carefully chosen) properties that we will have to accept in order to characterise those very entities and to be able to infer new properties of them.

This is the citation of Fano’s work that impressed so much Freudenthal. Fano followed strictly Segre’s programme. His main aim was to find “independent postulates” for projective geometry.

Following De Paolis he begins to build the “harmonic system”. But, he asks: if we have three points on a line, may we prove that the fourth harmonic is a new point, different from the three generating points?

His answer is no. He builds a model of geometry in which the axioms of projective geometry hold, while the fourth harmonic is one of the generating points.

In this geometry we have only seven points and seven lines, as shown by the following diagram.



7 Points: A, B, C, D, E, F, G

7 Lines: ABC, ADE, CDG, BDF, BGE, CEF, AGF

We have here the first example of finite geometry and the full development of an abstract point of view.

It is important however to notice how such a point of view was now taken for granted in the milieu of Italian mathematicians, so much that a young twenty-year-old did not even feel pressed to justify it in any particular way. By now, however, this totally abstract approach has become a common ground among the Italian mathematicians. In a way, Freudenthal had no reason to be amazed at recognising in Fano a precursor of Hilbert. For at least a decade such abstraction had been a topic of discussion among the Italian mathematicians, and in the Turin school of Segre and Peano in particular. Naturally, the question of the philosophical *origin* of the postulates is an altogether different matter. Just as, at least on a formal level, there is a difference between going from abstract objects (points), considered capable of numerous determinations, to the idea of *model* of a purely abstract structure, made of pure symbols.

10. Federigo Enriques

The problem of the philosophical origins of the postulates was one of the main interests of Federigo Enriques.

In his work of 1894 he wrote:

The direction that they (i. e. Fano and Amodeo) follow is quite different from the one we intend to follow, especially in that, while the clever authors set out to establish an arbitrary system of hypotheses capable of defining a linear space to which it is possible to apply the results of ordinary geometry, we try here to establish the postulates inferred from an experimental intuition of space which are easier to work with in order to define the object of projective geometry

As to those intuitive concepts, we do not intend to introduce anything other than their logical relation, so that a geometry thus founded can still be given an infinite number of

interpretations, where to its element called “point” an arbitrary meaning [our italic] is ascribed. We think that the experimental origin of geometry should not be forgotten while researching those very hypotheses on which it is founded

The same standing was expressed by Enriques in his influential book *Projective Geometry*.

Projective geometry can be considered as an abstract science and hence receive different interpretations from the intuitive one, assuming that its elements (points, lines, planes) are concepts determined in whichever way, and that entertain the logic relations expressed by the postulates

In 1902 Enriques' book was translated into German. Klein's foreword shows clearly the principal peculiarities of Italian approach:

Over the last two decades Italy has been the true centre of advanced research in the field of projective geometry. Among the specialists, this is well known.... But the Italian researchers have gone far beyond also on a practical level: they have not disdained to draw some didactic conclusions from their own studies. The remarkable textbooks for middle and secondary schools that have been born out of this attitude, can be made accessible to wider audiences by means of adequate translations. And this is all the more desirable in Germany since our didactic literature has lost all touch with recent research achievements. Therefore, both the translator and the publishing house that offer here a German translation of Enriques' projective geometry, can count from the very beginning on our unfailing support. ... We are not lacking in stimulating works that would be adequate for an introduction to projective geometry, but I do not know of any one in particular that offers a systematic construction of this theory, in an up-to-date form, and in an equally clear and exhaustive way. Moreover, the presentation is always intuitive, but completely rigorous, as it could only be after the clever researches on the foundations of projective geometry presented in earlier essays by the same author. I would like to draw particular attention to the presentation of the metric: the clear and explicit treatment of its foundation by means of the absolute (in the plane by means of a circle with a known centre)

I will not discuss here the axiomatic approach of Enriques, and I want to conclude my talk quoting Enriques who gives a very clear idea of Italian geometric school and the deep links between foundations of geometry and algebraic geometry:

Thanks to Klein and Lie, the concept of abstract geometry made great progress, and (after Segre) it became an ordinary tool for the contemporary Italian geometers. Indeed nothing is more fertile than the multiplication of our intuitive powers operated by this principle: it is as if besides the mortal eyes with which we examine a figure, we have

thousands of spiritual eyes that complete manifold transfigurations; all this while the unity of the object shines in our enriched reason, and it enables us to easily go from one form to another.

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